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CREEP AND STRESS RELAXATION IN A PLATE LOADED ALONG THE CONTOUR OF A CIRCULAR HOLE

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Analytical solutions of the creep and stress-relaxation boundary-value problems of a plate loaded externally along the contour of a circular hole are obtained using an unsteady creep model based on nonclassical representations for elastic and viscous properties of materials. It is assumed that one force component and one displacement component are specified at the boundary.

Key words: *unsteady creep, stress relaxation, plane stress state.*

1. Basic Relations of the Model. As was noted in [1], the equations of the classical theory of elasticity follow from the assumptions

$$e_{ij} = \frac{\partial U}{\partial \sigma_{ij}}, \qquad U = U_1(\sigma) + U_2(\Sigma) = \frac{3(1-2\nu)}{2E}\,\sigma^2 + \frac{1+\nu}{3E}\,\Sigma^2,$$
$$\sigma = \sigma_{kk}/3, \qquad \Sigma = \sqrt{3/2}\{(\sigma_1-\sigma)^2 + (\sigma_2-\sigma)^2 + (\sigma_3-\sigma)^2\}^{1/2}, \tag{1.1}$$

where e_{ij} and σ_{ij} are the components of the strain and stress tensors, respectively, σ_1, σ_2, and σ_3 are the principal values of the stress tensor, E is Young's modulus, and ν is Poisson's ratio. The specific quadratic dependence of the Gibbs potential U on the stress-tensor invariants σ and Σ yields the classical linear isotropic Hooke's law for an elastic medium. The stress-tensor invariant Σ is a homogeneous function of its arguments $\sigma_i - \sigma$. Ivlev [1] suggested to consider other possibilities of determining Σ with the homogeneity property being retained. An important role belongs here to the piecewise-linear relations

$$\Sigma = \max|\sigma_i - \sigma_j|; \tag{1.2}$$

$$\Sigma = (3/2)\max|\sigma_i - \sigma|. \tag{1.3}$$

These relations constrain the possible choice of Σ. Indeed, the surfaces $\Sigma = \text{const}$ determined by dependences (1.2) and (1.3) in the principal-stress space are hexagonal prisms with the generatrices parallel to the straight line $\sigma_1 = \sigma_2 = \sigma_3$. Their cross sections in the deviatoric plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ are shown in Fig. 1. The outer and inner hexagons are described by functions (1.3) and (1.2), respectively. It was shown in [2] that all possible convex surfaces $\Sigma = \text{const}$ should lie between these two prisms. Thus, dependences (1.2) and (1.3) are of interest as the limiting cases.

Bykovtsev and Yarushina [3] stated that creep of materials can be described using similar relations. They assumed that

$$e_{ij} = e^e_{ij} + e^v_{ij}, \tag{1.4}$$

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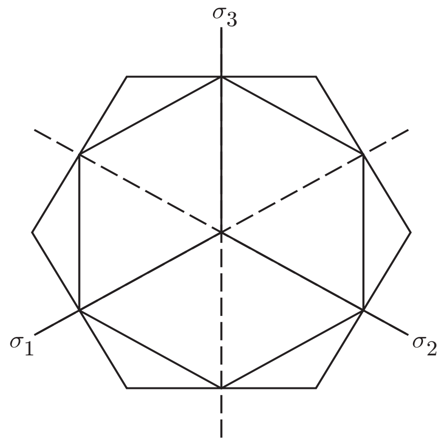

Fig. 1

where e_{ij} are the small total strains of the medium and e^e_{ij} and e^v_{ij} are the components of the elastic-strain and creep-strain tensors, respectively. The elastic strains were taken in the form (1.1) and the creep strains were given by

$$\dot{e}^v_{ij} = \frac{de^v_{ij}}{dt} = \frac{\partial V}{\partial \sigma_{ij}}, \qquad V = V(\Sigma). \tag{1.5}$$

In particular, the creep-rate potential can be written as

$$V = \frac{B}{n+1}\,\Sigma^{n+1}.$$

In this case, creep occurs in accordance with the Norton power law. The properties of the model relations are discussed in detail in [3, 4].

We consider an infinite thin plate with a circular hole of radius r_0 which is in equilibrium under the action of forces applied to the hole contour. The plane stress state of the plate is implied. Below, we consider the following boundary-value problems.

1. A rigid punch is pressed into the hole of the plate, which deforms in such a manner that the radial displacements of the contour are known and remain unchanged. Over the entire plate, stress relaxation occurs, and irreversible creep strains are accumulated. The shear stresses vanish on the contour:

$$u_r\Big|_{r=r_0} = u(\theta), \qquad \sigma_{r\theta}\Big|_{r=r_0} = 0. \tag{1.6}$$

Hereinafter, (r, θ) are polar coordinates and u_i are the displacement-vector components.

2. Under the action of tangential forces and a constant pressure applied to the hole, the circular hole is twisted so that the tangential displacement is known:

$$u_\theta\Big|_{r=r_0} = v(\theta), \qquad \sigma_{rr}\Big|_{r=r_0} = P = \text{const}. \tag{1.7}$$

In both cases, it is assumed that stresses in the entire plate satisfy the condition

$$\sigma_1 = -\sigma_2 < \sigma_3 = 0. \tag{1.8}$$

Moreover, it is assumed that there are no irreversible strains in the plate at the initial time $t = 0$. The temperature remains unchanged during the entire process. The forces acting at the contour are self-balanced. At infinity, the stresses and displacements vanish.

2. Derivation of the Resolving Equations of the Problem. Burenin and Yarushina [4] showed that, if the stress belongs to the edge of the piecewise-linear surface $\Sigma = \text{const}$, the plane stress state is statically determinable. They also studied the special features of the model relations at different facets and edges of the surface $\Sigma = \text{const}$.

We note that condition (1.8) implies that stresses belong to the edge of the piecewise-linear surface (1.3) formed by the intersecting facets

$$\Sigma = 3(\sigma - \sigma_1)/2, \qquad \Sigma = 3(\sigma_2 - \sigma)/2. \tag{2.1}$$

Since $\sigma_3 = 0$, Eqs. (2.1) can be considered as a system of two algebraic equations with two unknowns σ_1 and σ_2. Resolving this system, we obtain

$$\sigma_1 = -2\Sigma/3, \qquad \sigma_2 = 2\Sigma/3.$$

In this case, the stress-tensor components can be written as

$$\sigma_{11} = -\sigma_{22}, \qquad \sigma_{22} = (2\Sigma/3)\cos 2\varphi, \qquad \sigma_{12} = -(2\Sigma/3)\sin 2\varphi, \tag{2.2}$$

where φ is the angle between the first principal direction of the stress tensor and the Ox_1 axis. Thus, the number of independent static unknowns in the problem is reduced by one. By virtue of the first relation in (2.2), in the absence of body forces, the equations of equilibrium become

$$-\sigma_{22,1} + \sigma_{12,2} = 0, \qquad \sigma_{12,1} + \sigma_{22,2} = 0. \tag{2.3}$$

Conditions (2.3) are the Cauchy–Riemann conditions of analyticity of the function

$$f(z,t) = \sigma_{22} + i\sigma_{12} \tag{2.4}$$

of the complex variable $z = x_1 + ix_2$. All static quantities can now be expressed in terms of the stress function (2.4). As only two of them are independent, we write these equations only for Σ and φ:

$$\Sigma = 3|f(z,t)|/2, \qquad \cos 2\varphi - i\sin 2\varphi = f(z,t)/|f(z,t)|.$$

Using relations (1.1)–(1.5), we find the kinematic quantities of the problem:

$$e^e_{ij} = \frac{1}{3}\,U_1'(\sigma)\delta_{ij} + U_2'(\Sigma)\Big(\frac{\partial\Sigma}{\partial\sigma_1}\,l_il_j + \frac{\partial\Sigma}{\partial\sigma_2}\,m_im_j\Big), \qquad \dot{e}^v_{ij} = V'(\Sigma)\Big(\frac{\partial\Sigma}{\partial\sigma_1}\,l_il_j + \frac{\partial\Sigma}{\partial\sigma_2}\,m_im_j\Big) \tag{2.5}$$

(l_i and m_i are the direction cosines of the stress tensor). At the corner points of the surface $\Sigma = \text{const}$, which include the point of the actual stress state of the problem, the derivatives $\partial\Sigma/\partial\sigma_i$ are indeterminate. In this case, we use the generalized expression

$$\frac{\partial\Sigma}{\partial\sigma_i} = \alpha\,\frac{\partial\Sigma^{(1)}}{\partial\sigma_i} + (1-\alpha)\,\frac{\partial\Sigma^{(2)}}{\partial\sigma_i}, \qquad 0 \leqslant \alpha \leqslant 1, \tag{2.6}$$

where $\Sigma^{(1)} = \text{const}$ and $\Sigma^{(2)} = \text{const}$ are the surfaces that form the edge and α is a new unknown function responsible for the viscous-flow direction. It is worth noting that this definition of the derivative at the singular point on the stream surface $\Sigma = \text{const}$ is used in the plasticity theory [2].

Supplementing (2.5) by relations (2.6) and (1.4) and relations between the total strains and displacements, i.e.,

$$e_{ij} = (u_{i,j} + u_{j,i})/2,$$

and determining the displacements as functions of the complex variables $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$ and the time t, after elimination of the unknown function α, we obtain

$$u_1 - iu_2 = -\frac{3}{4}\int\limits_{z_0}^{z}\Big(\int\limits_0^t V'(\Sigma)\,\frac{f(z,t)}{|f(z,t)|}\,dt + U_2'(\Sigma)\,\frac{f(z,t)}{|f(z,t)|}\Big)\,dz + \psi(\bar{z},t). \tag{2.7}$$

The function $\psi(\bar{z},t)$ can be found from the boundary conditions. To determine α, by virtue of Eqs. (1.4), (2.5), and (2.6), we have

$$\begin{gathered}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = e^v_{11} + e^v_{22} + \frac{2}{3}\,U_1'(\sigma) + U_2'(\Sigma)\Big(\alpha - \frac{1}{2}\Big),\\ \dot{e}^v_{11} + \dot{e}^v_{22} = V'(\Sigma)(\alpha - 1/2).\end{gathered} \tag{2.8}$$

The function α should satisfy not only Eq. (2.8) but also the inequality $0 \leqslant \alpha \leqslant 1$, which ensures that the stresses belong to a chosen edge. If this inequality fails, the stresses belong to a facet of the piecewise-linear surface $\Sigma = \text{const}$ and, hence, one should replace Eq. (2.7) by other dependences that also follow from (2.5).

3. Solution of the Problems. Before solving each particular boundary-value problem formulated in Sec. 1, we make some general conclusions. To determine the stresses in the plate, it is convenient to expand the function $f(z,t)$ into the Laurent series in the neighborhood of an infinitely distant point:

$$f(z,t) = \sum_{m=-\infty}^{\infty} a_m(t) z^m. \tag{3.1}$$

Since it is assumed in both problems that no loads are applied to the plate at infinity, expansion (3.1) should contain only the regular part

$$f(z,t) = \sum_{k=1}^{\infty} a_k(t) z^{-k}. \tag{3.2}$$

The loads acting on the plate are assumed to be self-balanced, which implies that

$$-F_1 + iF_2 = \int_0^{2\pi} f(r\,\mathrm{e}^{i\theta}, t)\,\mathrm{e}^{i\theta}\, d\theta = \frac{2\pi a_1}{r} = 0, \tag{3.3}$$

where F_1 and F_2 are the components of the principal vector of external forces. It follows from Eq. (3.3) that $a_1 = 0$.

The boundary conditions (1.6) and (1.7) determine the stress-tensor and displacement-vector components in polar coordinates. We write these components as

$$\sigma_{\theta\theta} + i\sigma_{r\theta} = f(z,t)\,\mathrm{e}^{2\theta i}, \qquad \sigma_{rr} = -\sigma_{\theta\theta},$$

$$u_r - iu_\theta = (u_1 - iu_2)\,\mathrm{e}^{\theta i}. \tag{3.4}$$

To solve the problem subject to the boundary conditions (1.6), we first rewrite the boundary conditions with allowance for Eq. (3.4) as

$$\operatorname{Re}\{(u_1 - iu_2)\,\mathrm{e}^{i\theta}\}\Big|_{r=r_0} = u(\theta), \qquad \operatorname{Im} f(r_0\,\mathrm{e}^{i\theta}, t)\,\mathrm{e}^{2\theta i} = 0, \tag{3.5}$$

which implies that the Laurent series expansion (3.2) of the stress function is such that all coefficients are $a_k = 0$ except for $a_2 = P$, which is a real function of time. Thus,

$$f(z,t) = P(t)/z^2. \tag{3.6}$$

Integrating the relation for displacements (2.7) with allowance for (3.6), we obtain

$$u_1 - iu_2 = \frac{M}{\bar{z}^{n-1} z^n} \int_0^t |P|^{n-1} P\, dt + N\,\frac{P}{z} + \psi(\bar{z}, t),$$

where $M = (3/2)^{n+1} B/(2n)$ and $N = 3/(8\mu)$. Expansion of $\psi(\bar{z},t)$ into the Laurent series yields

$$\psi(\bar{z},t) = \sum_{k=1}^{\infty} b_k(t) \bar{z}^{-k}. \tag{3.7}$$

To determine the coefficients $b_k(t)$ of this expansion and the unknown function $P(t)$, we use the first condition of (3.5). On the contour of the circular hole, we have

$$u(\theta) = b_0 + \operatorname{Re} \sum_{k=1}^{\infty} \frac{b_k}{r_0^k}\,\mathrm{e}^{(k+1)\theta i}, \tag{3.8}$$

where b_0 is given by

$$b_0 = \frac{M}{r_0^{2n-1}} \int_0^t |P|^{n-1} P\, dt + N\,\frac{P}{r_0}. \tag{3.9}$$

Relation (3.8) is the Fourier series expansion of the real function. The coefficients of this expansion are determined by the formulas

$$b_0 = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} u(\theta)\, d\theta, \qquad b_k = \frac{r_0^k}{\pi} \int\limits_{-\pi}^{\pi} u(\theta)\, \mathrm{e}^{-(k+1)\theta i}\, d\theta, \quad k \geqslant 1.$$

Using (3.9), we obtain the unknown function $P(t)$:

$$P(t) = \pm \Big\{ \frac{M}{N} \frac{n-1}{r_0^{2n-2}}\, t + \Big| \frac{N}{b_0 r_0} \Big|^{n-1} \Big\}^{1/(1-n)}. \tag{3.10}$$

In (3.10), the plus refers to $b_0 > 0$, which is the case where a punch is pressed into a hole whose diameter is smaller than that of the punch; the minus refers to the case where the circular hole in the plate is stretched instantaneously over a rigid rod whose diameter is smaller than that of the hole ($b_0 < 0$). In both cases, the absolute values of stresses in the plate decreases, which shows that the solution constructed does describe the stress relaxation.

To determine the applicability limits of the solution constructed, we find the function α as the solution of system (2.8):

$$\alpha(R, \gamma, \theta) = \frac{1}{2} \mp \frac{3}{2} \frac{\gamma - 1}{\gamma} \frac{n-1}{n} (1 - R^\gamma) - \frac{3}{2} R^\gamma \operatorname{Re} \sum_{k=1}^{\infty} \frac{k b_k}{|b_0|} \Big(\frac{\gamma - 1}{n} \Big)^{(k-1)/(2n-2)} \frac{\mathrm{e}^{(k+1)\theta i}}{r_0^k}. \tag{3.11}$$

Here $R = NP/(r_0 b_0)$ and $\gamma = n(r_0/r)^{2n-2} + 1$. In (3.11), the minus and plus refer to the cases with $b_0 > 0$ and $b_0 < 0$, respectively.

Below, we confine our analysis to the solution where b_0 and b_1 are real numbers and other coefficients are $b_k = 0$. In this case, the hole becomes an ellipse with the semiaxes

$$a = r_0 + b_0 + b_1/r_0, \qquad b = r_0 + b_0 - b_1/r_0.$$

The inequality $0 \leqslant \alpha \leqslant 1$, which ensures the adequacy of the constructed solution, is satisfied in the entire plate provided that

$$\frac{\gamma - 1}{\gamma} \frac{n-1}{n} (1 - R^\gamma) + \frac{b_1}{r_0 |b_0|} R^\gamma \leqslant \frac{1}{3}, \tag{3.12}$$

as it follows from (3.11). We note that $0 < R \leqslant 1$ and $1 < \gamma \leqslant n + 1$. For $R = 1$, i.e., at the initial time, inequality (3.12) is valid if $3b_1 \leqslant r_0 |b_0|$, which means that the semiaxes of the ellipse pressed should satisfy the inequality

$$\Big| \frac{a - b}{a + b - 2r_0} \Big| \leqslant \frac{1}{3}.$$

The expression in the left side (3.12) reaches the extremum for $\gamma = n + 1$, i.e., at the edge of the hole; therefore, inequality (3.12) is satisfied in the entire plate for $R^* \leqslant R \leqslant 1$, where

$$R^* = \Big\{ \frac{2}{3} \frac{(2 - n) r_0 |b_0|}{b_1 (n+1) - r_0 |b_0| (n - 1)} \Big\}^{1/(n+1)}. \tag{3.13}$$

This implies that the plate behavior is described by the constructed solution until a certain time t^*, which can be readily found from relations (3.10) and (3.13). For $t > t^*$, the stress state cannot correspond to the chosen edge in the entire region.

To construct the solution of the problem with the boundary conditions (1.7), we rewrite its boundary conditions as

$$\operatorname{Im} \{ (u_1 - i u_2)\, \mathrm{e}^{i\theta} \} \Big|_{r = r_0} = -v(\theta), \qquad \operatorname{Re} \{ f(r_0\, \mathrm{e}^{i\theta}, t)\, \mathrm{e}^{2\theta i} \} = -P. \tag{3.14}$$

Expanding the function $f(z, t)$ into the Laurent series (3.2), by virtue of (3.14), we obtain

$$f(z, t) = \{ -P + i\tau(t) \} r_0^2 / z^2.$$

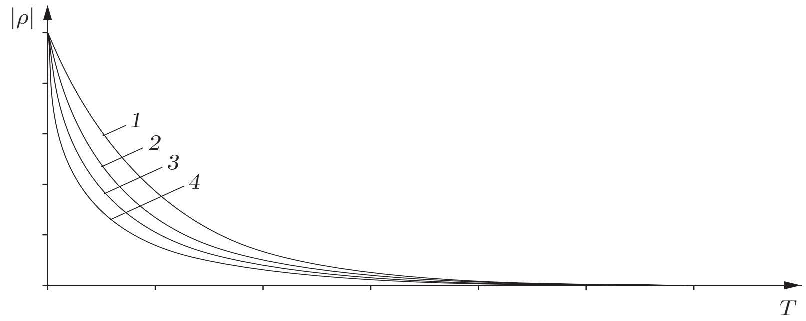

Fig. 2

In this case, the equation for displacements (2.7) becomes

$$u_1 - iu_2 = M\,\frac{r_0^{2n}}{z^n \bar{z}^{n-1}} \int\limits_0^t (P^2+\tau^2)^{(n-1)/2}(-P+i\tau)\,dt + N\,\frac{r_0^2}{z}\,(-P+i\tau) + \psi(\bar{z},t).$$

Using expansion (3.7) for $\psi(\bar{z},t)$, we find

$$b_0 = Mr_0 \int\limits_0^t (P^2+\tau^2)^{(n-1)/2}\tau\,dt + Nr_0\tau = -\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} v(\theta)\,d\theta,$$

$$b_k = -\frac{r_0^k}{\pi}\int\limits_{-\pi}^{\pi} v(\theta)\,\mathrm{e}^{(k+1)\theta i}\,d\theta. \tag{3.15}$$

The unknown shear stress $\tau(t)$ can be determined from the first equation of (3.15). In the case where $P = 0$, i.e., the hole contour is free from compressive stresses, we obtain

$$\tau(t) = \pm\Big\{\Big|\frac{b_0}{Nr_0}\Big|^{1-n} + \frac{M}{N}\,(n-1)t\Big\}^{1/(1-n)},$$

where the plus and minus refer to $v(\theta) < 0$ and $v(\theta) > 0$, respectively.

For $P \neq 0$, the solution of Eq. (3.15) can be written as

$$T = \ln\frac{\rho_0}{\rho} + \frac{n-1}{4}\sum_{k=0}^{\infty}\frac{\Gamma^2(k+1)}{\Gamma(1)}\,\frac{\Gamma^2(2)}{\Gamma(k+2)}\,\frac{\Gamma^2((n+1)/2+k)}{\Gamma((n+1)/2)}\,\frac{(-1)^k}{k!}\,(\rho^{2k+2}-\rho_0^{2k+1}), \tag{3.16}$$

where $\Gamma(k) = \int\limits_0^{\infty} \mathrm{e}^{-x}\,x^{k-1}\,dx$ is the gamma function. Equation (3.16) contains the dimensionless parameters

$$T = P^{n-1}tM/N, \qquad \rho = \tau(t)/P, \qquad \rho_0 = b_0/(Nr_0P).$$

Figure 2 shows the curves of the shear-stress relaxation on the hole contour; curves 1, 2, 3, and 4 refer to $n = 1$, 3, 6, and 8, respectively.

Finally, the direction of the normal to the stream surface $\Sigma = \text{const}$ is determined by

$$\alpha(\rho,\xi,\theta) = \frac{1}{2} + \frac{3}{2}\,\frac{n-1}{n}\,\frac{1}{\sqrt{1+\rho^2}}\Big(1-\Big(\frac{\rho}{\rho_0}\Big)^{n\xi^{2n-2}}\Big) - \frac{3}{2}\,\frac{\rho_0}{b_0}\,\frac{1}{\sqrt{1+\rho^2}}\Big(\frac{\rho}{\rho_0}\Big)^{n\xi^{2n-2}}\,\mathrm{Re}\sum_{k=1}^{\infty}\frac{k\xi^{k-1}b_k}{r_0^k}\,\mathrm{e}^{(k+1)\theta i}, \tag{3.17}$$

where $\xi = r_0/r$. At the initial time $t = 0$, we have

$$\alpha = \frac{1}{2} - \frac{3}{2}\,\frac{\rho_0}{\sqrt{1+\rho_0^2}}\,\mathrm{Re}\sum_{k=1}^{\infty}\frac{k\xi^{k-1}b_k}{b_0r_0^k}\,\mathrm{e}^{(k+1)\theta i}.$$

As $0 \leqslant \alpha \leqslant 1$, the stresses in the plate correspond to the chosen edge if

$$\frac{\rho_0}{\sqrt{1+\rho_0^2}} \sum_{k=1}^{\infty} \frac{k}{r_0^k} \frac{|b_k|}{|b_0|} \leqslant \frac{1}{3}. \tag{3.18}$$

By virtue of Eq. (3.15), this inequality determines which displacements $v(\theta)$ should be specified on the boundary. For $t > 0$ and $0 < \xi \leqslant 1$, the following constraint, which follows from (3.17), should be valid:

$$\frac{n-1}{n}\Big(1 - \Big(\frac{\rho}{\rho_0}\Big)^{n\xi^{2n-2}}\Big) + \frac{\rho_0}{\sqrt{1+\rho_0^2}} \Big(\frac{\rho}{\rho_0}\Big)^{n\xi^{2n-2}} \sum_{k=1}^{\infty} \frac{k\xi^{k-1}}{r_0^k} \frac{|b_k|}{|b_0|} \leqslant \frac{1}{3}. \tag{3.19}$$

The right side in Eq. (3.19) increases with increasing ξ and decreasing ρ. It reaches the maximum at the hole boundary where $\xi = 1$. In this case, there exists a value of ρ^* with which (3.18) becomes an equality. For $\rho^* \leqslant \rho \leqslant \rho_0$ or, equivalently, for $0 \leqslant t \leqslant t^*$, the inequality $0 \leqslant \alpha \leqslant 1$ is satisfied in the entire plate. For $t > t^*$, the relations given above fail to describe the behavior of the plate.

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